

# *Forecasting Volatility of Hang Seng Index and its Application on Reserving for Investment Guarantees*

## Abstract

In this paper, we study the predictability of Hang Seng Index daily volatility and develop a daily volatility forecasting approach for reserving for investment guarantees in Hong Kong.

Generalized autoregressive conditional heteroskedasticity (GARCH) model for daily volatility forecasting is investigated. Hang Seng Index volatilities and returns from 1999 to 2004 are fitted into the model to generate forecasted daily volatilities. These forecasted daily volatilities are then investigated by the value-at-risk approach and validated by the binomial test. It is found that GARCH model has satisfactory model fitting ability and daily volatility forecasting results.

Forecasted daily volatilities from GARCH model could be used for reserving for investment guarantees. By simulations using a random generator and forecasted daily volatilities, the most adverse Hang Seng Index price with a defined time frame could be estimated with a pre-defined level of confidence. It is of great values to provision for liabilities arising from Hang Seng Index embedded investment guarantees.

# Contents

1. Introduction	p.3
2. Hang Seng Index Return and Volatility Data	p.4
2.1 Daily Return, Overnight Return and 5-minute Return	p.5
2.2 Historical Volatility	p.6
2.3 Realized Volatility	p.6
2.4 Implied Volatility	p.8
2.5 Descriptive Statistics	p.8
3. Model Estimation	p.13
4. Forecasting Methodology	p.14
4.1 Rolling Window	p.14
4.2 Value-at-Risk Approach	p.15
4.3 Binomial Test	p.16
5. Empirical Results	p.17
5.1 Parameter Estimation	p.17
5.2 Forecasting Results	p.20
6. Reserving by Simulation	p.23
7. Conclusion and Further Research	p.24
References	p.25

# 1 Introduction

Insurers consider asset's volatility as one of the main criteria when reserving for investment guarantees. Both the Canadian Institute of Actuaries (2002), the American Academy of Actuaries (2001) and the Actuarial Society of Hong Kong (2001) published guidelines or reports on stochastic approaches to reserving for investment guarantees. Volatility modeling in financial market therefore gains much attention in the insurance industry.

Early empirical studies of volatility modeling and forecasting can be dated back to the 1980's. Engle (1982) developed models which capture time-varying volatility process. Since then, historical volatility, realized volatility and implied volatility have been commonly used in volatility forecasting studies. Forecasting performance of historical volatility models has been confirmed by Bollerslev (1986) and Shephard (1996). Positive results are found in the empirical studies of forecasting performances of realized volatility models by Barndorff-Nielsen and Shephard (2002). Empirical studies by Christensen and Prabhala(1998) and Fleming (1998) also indicated implied volatility as a good predictor of daily volatility.

In this paper, we investigate the generalised autoregressive conditional heteroskedasticity (GARCH) model developed by Bollerslev (1986) for Hang Seng Index daily volatility forecasting. We also incorporate realized volatility as an explanatory variable in the variance equation of the GARCH model as suggested by Martens (2001). Another possible explanatory variable could be implied volatility or historical volatility, see Hol, Jungbacker, and Koopman (2005).

The empirical study is for Hang Seng Index series over the period 15 February 1999 to 20 April 2004. Estimation results, including maximum likelihood estimators and akaike information criterion values, are reported for the whole sample. Forecasting results generated by 'rolling window' are also presented. The forecasting

performance of the GARCH model for the last 500 transaction days is investigated by the value-at-risk approach and validated by the binomial test. It will be concluded that the GARCH model has satisfactory daily volatility forecasting performance. Based on simulations, forecasted Hang Seng Index volatilities from the GARCH model are then applied to reserving for investment guarantees.

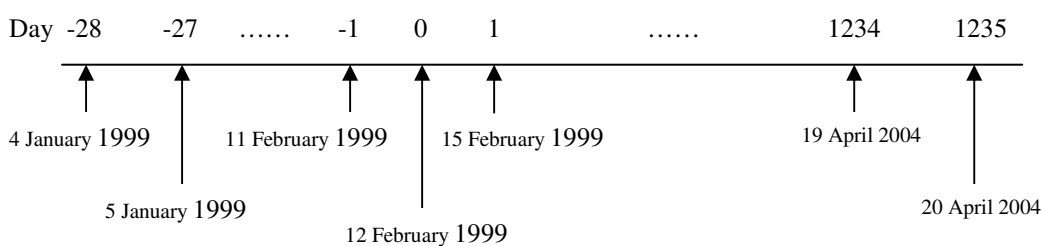
This paper is organized as follows. In the next section, we present the volatility measurements used in this empirical study and summarize their statistical properties. In section 3, a framework for the GARCH model is developed. Details about forecasting methodology are presented in section 4. Estimation and forecasting results follow in section 5. Section 6 discusses the application of the forecasted daily volatilities on reserving for investment guarantees. Section 7 concludes.

## 2 Hang Seng Index Return and Volatility Data

The data of this paper consist of Hang Seng Index transaction prices between 4 January 1999 and 20 April 2004. There are 1265 transaction days within this period. Data before 1999 are not included in this paper mainly due to the Asian financial crisis in 1997 and 1998. In these 2 years, Hang Seng Index fluctuated a lot due to unexpected environmental and economic factors. Including data in these two years may affect model's forecasting performance.

We denote 15 February 1999 as day 1, 19 February 1999, which is the following transaction day, as day 2 and so on. 20 April 2004 is denoted as day 1235. Dates before 15 February 1999 are denoted as non positive integers because data before that day are not used for model fitting and volatility forecasting. They are only used to construct historical volatilities on or after 15 February 1999.

Chart 1: Timeline showing the analogy between date and day.



## 2.1 Daily Return, Overnight Return and 5-minute Return

Daily return of Hang Seng Index is the log difference between the closing prices of Hang Seng Index on consecutive transaction days. The daily return at day  $n$  is

$$R_n = 100 (\ln P_n - \ln P_{n-1}) \quad n = -28, \dots, 1235 \quad (1)$$

where  $P_n$  is the closing price of Hang Seng Index at day  $n$ .

In order to calculate 5-minute return, price at this moment and price 5 minutes ago are recorded. The 5-minute return at this moment is the log difference between these 2 prices, that is

$$R_{n,d} = 100 (\ln P_{n,d} - \ln P_{n,d-1}) \quad n = 1, \dots, 1235 \quad d = 1, \dots, 55 \quad (2)$$

where  $P_{n,d}$  is the price at trading day  $n$ , at the 5 minute mark  $d$ .  $P_{n,0}$  is the opening price at day  $n$ .

The Hong Kong Stock Exchange usually opens from 9:45am to 12:30nn and from 2:30pm to 4:15pm. For each normal transaction day, there are 56 5-minute prices, resulting in 55 5-minute returns. However, not all transaction days have 55 5-minute returns. For example, the Hong Kong Stock Exchange only opens for the morning during Christmas Eve.

Overnight return of Hang Seng Index at day  $n$  is the log difference between opening price at day  $n$  and the closing price at day  $n-1$ . The overnight return at day  $n$  is

$$R_{n,0} = 100 (\ln P_{n,0} - \ln P_{n-1,55}) \quad n = 1, \dots, 1235 \quad (3)$$

where  $P_{n,0}$  is the opening price at day n and  $P_{n-1,55}$  is the closing price at day n-1.

## 2.2 Historical Volatility

Historical volatility is the variance of daily returns in the preceding 30 consecutive transaction days. The historical volatility at day n is

$$\bar{\sigma}_n^2 = \frac{1}{29} \sum_{m=n-29}^n (R_m - \bar{R}_n)^2 \quad n=1, \dots, 1235 \quad (4)$$

where  $\bar{R}_n = \frac{1}{30} \sum_{m=n-29}^n R_m$ . In section 2.1, we have 1264 daily returns, resulting in 1235

historical volatilities by using equation (4). The calculated historical volatilities are from 15 February 1999 to 20 April 2004 (from day 1 to day 1235).

## 2.3 Realized Volatility

Realized volatility is generally acknowledged as the variance of 5-minute returns within a day. If we were to construct realized volatility using this concept, realized volatility at day n was

$$\tilde{\sigma}_n^2 = \left[ \frac{1}{54} \sum_{d=1}^{55} (R_{n,d} - \bar{\bar{R}}_n)^2 \right] \times 55 \quad n=1, \dots, 1235 \quad (5)$$

where  $\bar{\bar{R}}_n$  is the mean of 5-minute returns at day n. In this paper, we ignore opening price and closing price when constructing realized volatility because opening price and closing price tend to fluctuate a lot due to noises in the market. So realized volatility at day n becomes

$$\tilde{\sigma}_n^2 = \left[ \frac{1}{52} \sum_{d=2}^{54} (R_{n,d} - \bar{\bar{R}}_n)^2 \right] \times 55 \quad n=1, \dots, 1235 \quad (6)$$

However, equation (6) ignores the effect of overnight return, which also contributes some volatility to realized volatility. In order to account for the effect of overnight return, we adjust equation (6) to

$$\tilde{\sigma}_n^2 = \frac{\hat{\sigma}_{oc}^2 + \hat{\sigma}_{co}^2}{\hat{\sigma}_{oc}^2} \left[ \frac{1}{52} \sum_{d=2}^{54} (R_{n,d} - \bar{R}_n)^2 \right] \times 55 \quad n=1, \dots, 1235 \quad (7)$$

where “open to close” sample variance  $\hat{\sigma}_{oc}^2$  and “close to open” sample variance  $\hat{\sigma}_{co}^2$  are defined as

$$\hat{\sigma}_{oc}^2 = \frac{1}{1234} \sum_{n=1}^{1235} (R_{oc,n} - \bar{R}_{oc})^2$$

$$\hat{\sigma}_{co}^2 = \frac{1}{1234} \sum_{n=1}^{1235} (R_{co,n} - \bar{R}_{co})^2$$

$$R_{oc,n} = 100(\ln P_{n,55} - \ln P_{n,0})$$

$$R_{co,n} = 100(\ln P_{n,0} - \ln P_{n-1,55})$$

$\bar{R}_{oc}$  is the sample mean of “open to close” return  $R_{oc,n}$  and  $\bar{R}_{co}$  is the sample mean of “close to open” return  $R_{co,n}$ .  $\hat{\sigma}_{oc}$  represents the weight given to the volatility within a day and  $\hat{\sigma}_{co}$  represents the weight given to the volatility contributed by overnight return. One would easily recognize that  $\hat{\sigma}_{oc} + \hat{\sigma}_{co}$  represents the weight given to the total volatility.

In fact, one cannot guarantee the accuracy of the adjustment. Realized volatility obtained by equation (7) may be biased upwards or downwards. Section 3 shall explain how the GARCH model adjusts the bias by the model estimator.

Realized volatilities are constructed for the period 15 February 1999 to 20 April 2004 (from day 1 to day 1235) using equation (7). As mentioned in section 2.1, not all transaction days have 55 5-minute returns. For those transaction days with less than 55 5-minute returns, realized volatility is still constructed as the variance of 5-minute returns within that day, ignoring the opening price and closing price.

## 2.4 Implied Volatility

Based on Black Scholes model, implied volatility can be obtained from option pricing data whose underlying asset is Hang Seng Index. However, in real practice, constructing implied volatility is not simple. In each transaction day, there are many options whose underlying asset is Hang Seng Index, with different striking prices and time to maturity, leading to different values of implied volatility. In this paper, we obtain the dataset of implied volatility from Hong Kong Exchange and Clearing Limited. We obtain implied volatility  $\ddot{\sigma}_n^2$  on a daily basis for the period 15 February 1999 to 20 April 2004 (from day 1 to day 1235), where  $n=1, \dots, 1235$ .

## 2.5 Descriptive Statistics

The summary statistics for daily return, historical volatility, realized volatility and implied volatility are presented in Table 1.

Table 1: Summary of descriptive statistics for return and volatility series from 15 February 1999 to 20 April 2004 (from day 1 to day 1235)

	Mean	Stand. Dev.	Skewness	Kurtosis	Minimum	Maximum
Daily Return						
$R_n$	0.0306	1.7302	-0.03	1.95	-8.9026	6.7273
Squared Daily Return						
$R_n^2$	2.992	5.936	5.93	55.15	0	79.256
Historical Volatility						
$\tilde{\sigma}_n^2$	3.0378	1.8570	1.19	1.02	0.6242	10.0674
Realized Volatility						
$\hat{\sigma}_n^2$	2.5837	2.5207	3.16	14.8	0.1467	22.5719
Implied Volatility						
$\ddot{\sigma}_n^2$	2.8505	1.5193	1.34	3.07	0.4000	12.7238

The small magnitude of the skewness coefficient for daily return reflects the symmetry of daily return distribution. Both historical volatility, realized volatility and implied volatility have positive skewness and are skewed to the right, as demonstrated in figure 2. This may be explained by 2 volatile periods which occurred in the first quarter of 2000 (from day 200 to day 300) and near the end of the third quarter in 2001 (from day 620 to day 650). During these 2 volatile periods, daily returns fluctuated a lot.

The mean of historical volatility is close to the mean of implied volatility. However, mean of realized volatility is lower than that of historical and implied volatility, confirming the existence of bias in constructing realized volatility as mentioned in section 2.3.

Figure 1 presents the time series plot of daily return, historical, realized and implied volatility. High degree of fluctuation of daily return tends to increase all three volatility measurements, and vice versa. This supports the argument that volatility measurements have some value in measuring volatility. It is not surprising that historical volatility has several days lag when it responds to fluctuation in daily return. Based on the way it is constructed, historical volatility at day  $n$  is actually measuring volatility over the preceding 30 transaction days.

Sample autocorrelation functions (correlograms) are presented in figure 3. Both historical volatility, realized volatility and implied volatility demonstrate property of volatility persistency. It is within our expectations that historical volatilities are highly correlated with each other. In fact, they should be correlated with each other by the way they are constructed. Historical volatility at day  $n$  is constructed by daily returns from day  $n-29$  to day  $n$ . Historical volatility at day  $n+1$  is constructed by daily returns from day  $n-28$  to day  $n+1$ . Therefore historical volatilities at day  $n$  and day  $n+1$  should be highly correlated.

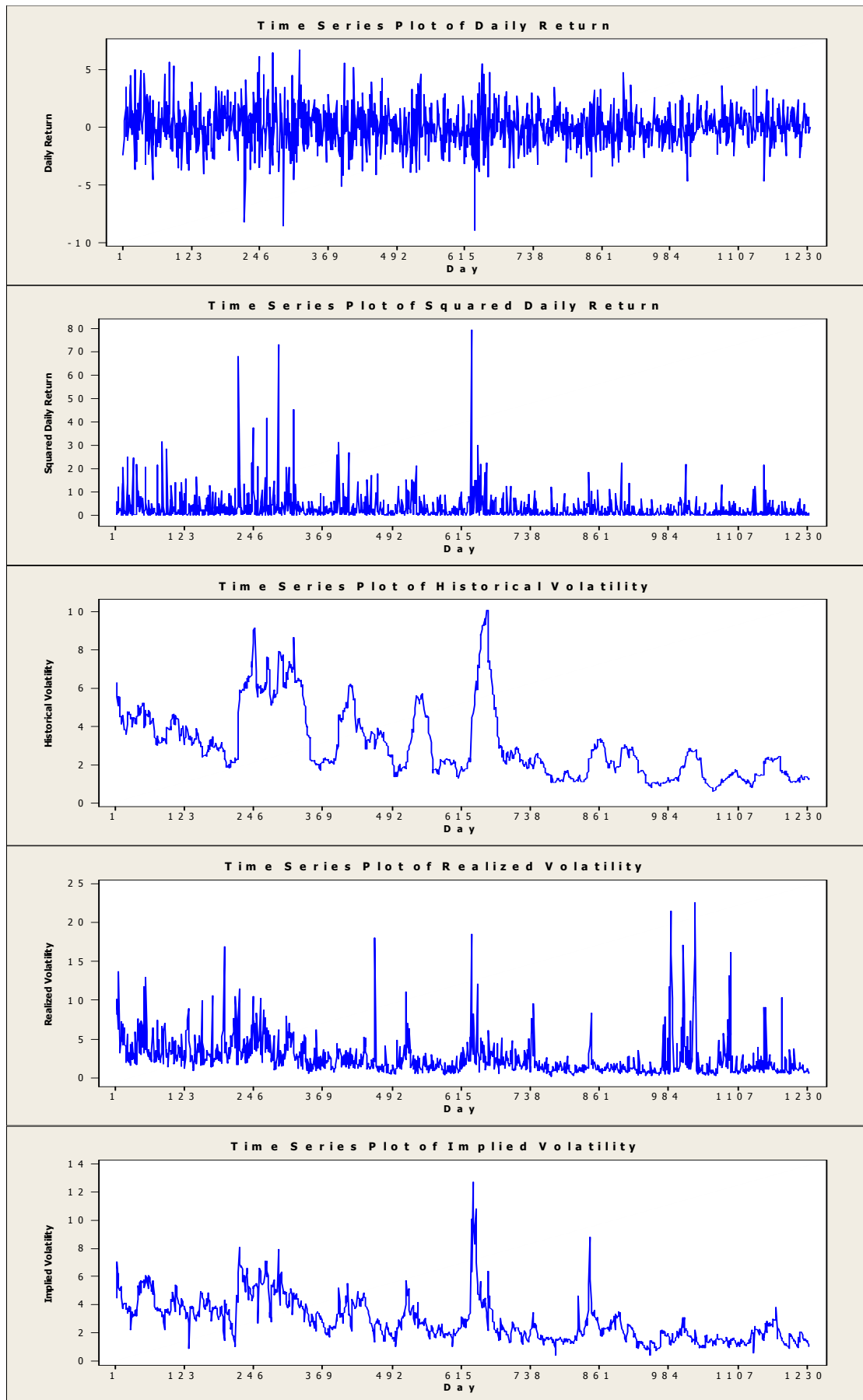


Figure 1: Daily time series plot of daily return  $R_n$ , squared daily return  $R_n^2$ , historical volatility  $\hat{\sigma}_n^2$ , realized volatility  $\tilde{\sigma}_n^2$  and implied volatility  $\ddot{\sigma}_n^2$

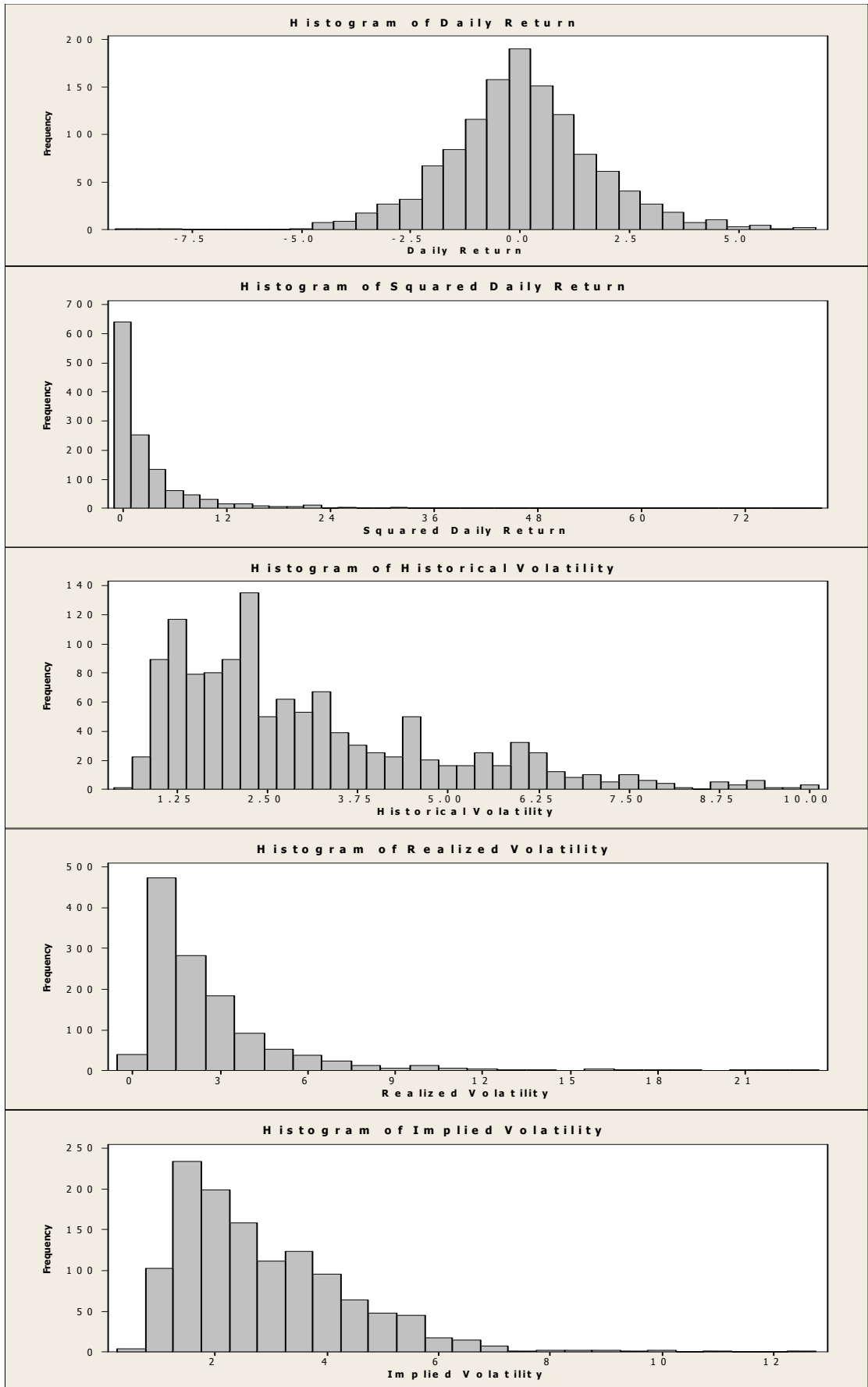


Figure 2: Histograms of daily return  $R_n$ , squared daily return  $R_n^2$ , historical volatility  $\tilde{\sigma}_n$ , realized volatility  $\tilde{\sigma}_n^2$  and implied volatility  $\ddot{\sigma}_n^2$

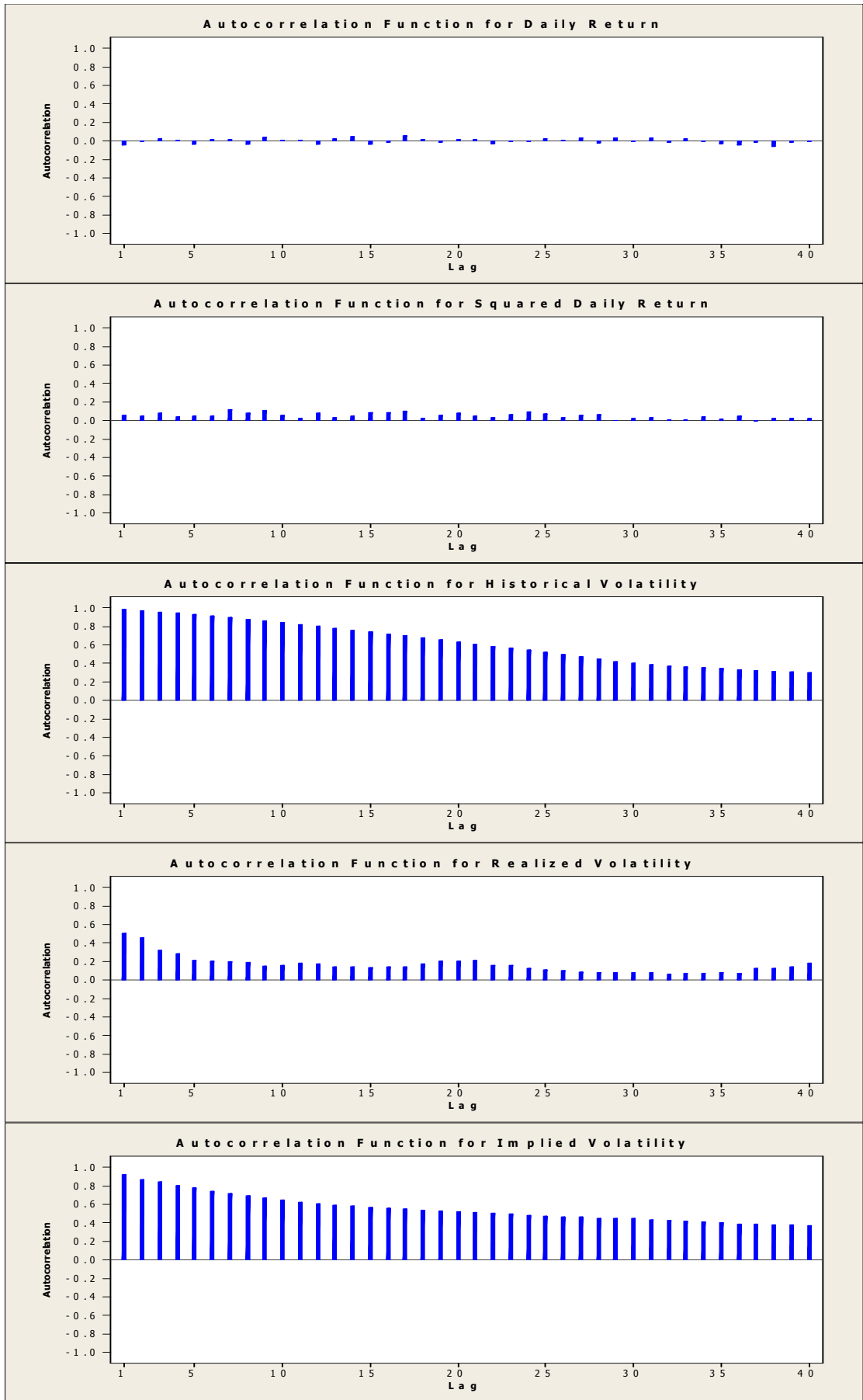


Figure 3: Correlograms of daily return  $R_n$ , squared daily return  $R_n^2$ , historical volatility  $\check{\sigma}_n^2$ , realized volatility  $\tilde{\sigma}_n^2$  and implied volatility  $\check{\sigma}_n^2$

### 3 Model Estimation

We are going to investigate the GARCH model for estimation. The form of a simple GARCH model is

$$\begin{aligned} R_n &= \mu + \sigma_n \varepsilon_n \\ \sigma_n^2 &= \omega + \alpha R_{n-1}^2 + \beta \sigma_{n-1}^2 \end{aligned} \quad \varepsilon_n \sim NID(0,1) \quad n=2, \dots, 1235 \quad (8)$$

where  $R_n$  is the daily return at day  $n$ ,  $\sigma_n^2$  is the daily volatility at day  $n$ ,  $\mu$ ,  $\omega$ ,  $\alpha$ , and  $\beta$  are parameters to be estimated with restrictions

$$\omega > 0 \quad \alpha \geq 0 \quad \beta \geq 0 \quad \alpha + \beta < 1$$

$\alpha + \beta$  measures daily volatility persistency. Based on model assumption, return at day  $n$  follows normal distribution with mean  $\mu$  and variance  $\omega + \alpha R_{n-1}^2 + \beta \sigma_{n-1}^2$ , which is conditional on past observations.

$$R_n | R_1, \dots, R_{n-1} \sim NID(\mu, \omega + \alpha R_{n-1}^2 + \beta \sigma_{n-1}^2)$$

The simple GARCH model can also be extended by adding explanatory variable into the variance equation so that equation (8) becomes

$$\begin{aligned} R_n &= \mu + \sigma_n \varepsilon_n \\ \sigma_n^2 &= \omega + \alpha R_{n-1}^2 + \beta \sigma_{n-1}^2 + \gamma s_{n-1}^2 \end{aligned} \quad \varepsilon_n \sim NID(0,1) \quad (9)$$

where  $s_{n-1}^2$  could be either historical volatility, realized volatility or implied volatility at day  $n-1$  and  $\gamma$  is a parameter to be estimated. Parameters in GARCH model are estimated by the maximum likelihood method. Exact computations can be done using SAS 9.1 TS Level 1M2.

For the GARCH model with realized volatility as an explanatory variable, there is no need to adjust the bias mentioned in section 2.3 as the estimator  $\hat{\gamma}$  automatically adjusts itself for the bias when data are fitted into the model.

For the simple GARCH model, the one day ahead daily volatility forecast at time N is computed as

$$\hat{\sigma}_{N+1}^2 = \hat{\omega} + \hat{\alpha}R_N^2 + \hat{\beta}\hat{\sigma}_N^2 \quad (10)$$

where  $\hat{\omega}, \hat{\alpha}, \hat{\beta}$  are maximum likelihood estimators of  $\omega, \alpha, \beta$  respectively and  $\hat{\sigma}_N^2$  is the estimated volatility at day N.

For the GARCH model with explanatory variable  $s_{n-1}^2$ , the one day ahead daily volatility forecast at time N is computed as

$$\hat{\sigma}_{N+1}^2 = \hat{\omega} + \hat{\alpha}R_N^2 + \hat{\beta}\hat{\sigma}_N^2 + \hat{\gamma}s_N^2 \quad (11)$$

where  $\hat{\gamma}$  is the maximum likelihood estimator of  $\gamma$ .

## 4 Forecasting Methodology

### 4.1 Rolling Window

The “rolling window” method is adapted for volatility forecasting. Dataset in this paper includes daily returns, historical, realized and implied volatilities in 1235 transaction days. Data from 1235 transaction days are divided into 500 samples. The first sample contains data from day 1 to day 735. Forecast of daily volatility at day 736 is generated by the model estimation based on the first sample. The second sample contains data from day 2 to day 736. Forecast of daily volatility at day 737 is generated by the model estimation based on the second sample. We repeat the above steps for 500 times and the 500<sup>th</sup> sample contains data from day 500 to 1234. Forecast of daily volatility at day 1235 is generated by the model estimation based on the 500<sup>th</sup> sample. Finally we obtain 500 forecasted daily volatilities, beginning from day 736 to day 1235.

## 4.2 Value-at-Risk Approach

One direct approach to validate the accuracy of forecasted daily volatilities is to compare them with the actual volatilities. However, actual volatilities are not observable. Value-at-risk approach is therefore used for validating accuracy of forecasted daily volatilities.

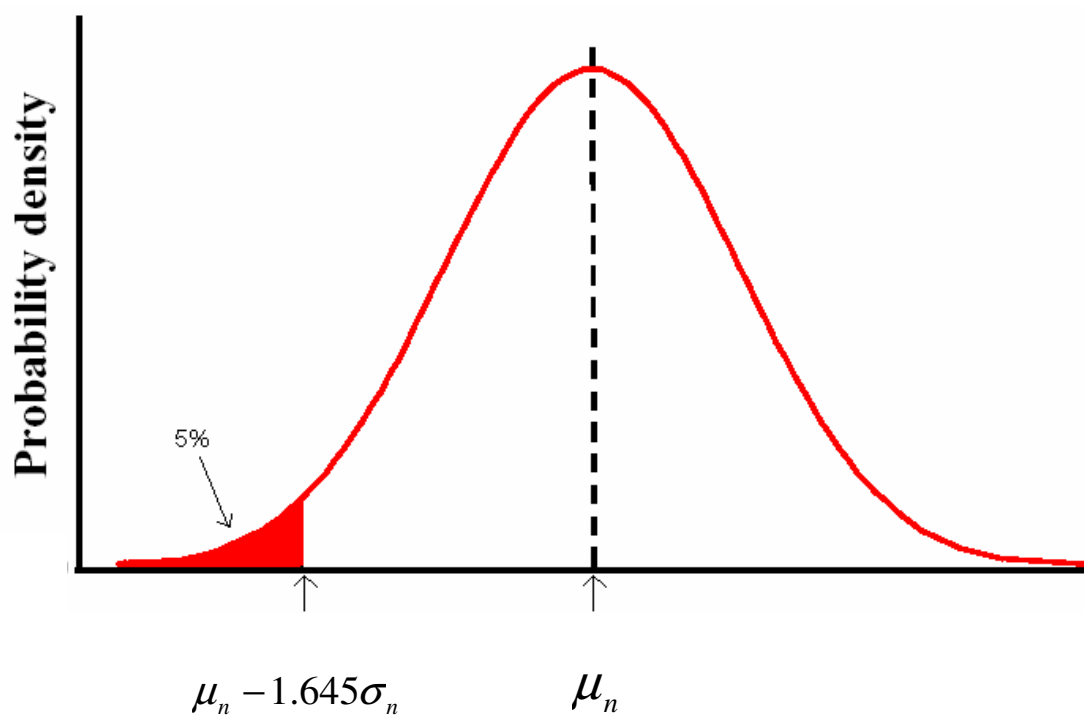
Returns are assumed to be normally distributed with different means and variances. That is, daily return at day  $n$ ,  $R_n$ , is assumed to be normally distributed with mean  $\mu_n$  and variance  $\sigma_n^2$ , i.e.

$$R_n \sim N(\mu_n, \sigma_n^2) \quad (12)$$

With reference to figure 4, daily return at day  $n$  has 0.05 probability to satisfy condition (13) and fall into the grey area.

$$R_n < \mu_n - 1.645\sigma_n \quad (13)$$

Figure 4: Probability Density Distribution of Normal Distribution



The concept of value-at-risk approach is as follows. If we assume forecasted daily volatility and estimated mean at day  $n$  are accurate and replace  $\sigma_n$  and  $\mu_n$  with  $\hat{\sigma}_n$  and  $\hat{\mu}_n$  respectively, return at day  $n$  should have 0.05 probability to satisfy condition (14).

$$R_n < \hat{\mu}_n - 1.645\hat{\sigma}_n \quad (14)$$

In other words, if return at day  $n$  has around 0.05 probability to satisfy condition (14), we can conclude that forecasted daily volatility and estimated mean at day  $n$  are accurate.

Now we have 500 forecasted daily volatilities. If we assume forecasted daily volatility and estimated mean at day  $n$  are accurate and replace  $\sigma_n$  and  $\mu_n$  with  $\hat{\sigma}_n$  and  $\hat{\mu}_n$  respectively for  $n=736, \dots, 1235$ , there should be around 25, i.e. 5% of 500, of them satisfying condition (14). In other words, if we observe around 25 returns satisfy condition (14), we can conclude that those 500 forecasted daily volatilities and estimated means are accurate.

We assume the mean of return  $\mu_n$  is the same for  $n=736, \dots, 1235$  and is the estimated parameter  $\hat{\mu}$  by fitting data from day 736 to day 1235.

### 4.3 Binomial Test

Binomial test is applied to find the range of number of returns satisfying condition (14) so that we fail to reject the hypothesis that those 500 forecasted daily volatilities and estimated means are accurate. Before applying the binomial test, we have to assume that daily returns are independent. This assumption is reasonable because daily returns have low correlations with each other, as demonstrated in figure 2. This assumption is also supported by the efficient market hypothesis.

$H_0$  is the null hypothesis that forecasted daily volatilities and estimated mean are accurate and  $H_1$  is the alternative hypothesis that forecasted daily volatilities and estimated mean are not accurate.

Under  $H_0$ , the probability that return at day  $n$  satisfies condition (14) is 0.05, i.e.  $p=0.05$ .  $n=500$  as there are 500 returns. Test statistics is the actual number of returns that satisfy condition (14). When significance level  $\alpha =0.05$ , lower bound =  $np - 1.96\sqrt{np(1-p)} =15.4$  and upper bound =  $np + 1.96\sqrt{np(1-p)} =34.6$ .

Out of the 500 returns, if the observed number of returns satisfying condition (14) is between 15.4 and 34.6, we fail to reject  $H_0$  at 5% significance level and conclude that forecasted daily volatilities and estimated mean are accurate.

The wide range of testing interval may be explained by the small amount of data used for value-at-risk approach and binomial test. The reason for including data from only 1235 transaction days is explained in the first paragraph of section 2.

It should be noted that rejecting  $H_0$  does not necessarily mean forecasted volatilities are not accurate. It may be the estimation errors of mean  $\hat{\mu}_n$  that results in the rejection decision. However, failure to reject  $H_0$  supports the conclusion that forecasted volatilities are accurate.

## 5 Empirical Results

### 5.1 Parameter Estimation

The whole dataset, which includes 1235 data for the period 15 February 1999 to 20 April 2004, is fitted into the GARCH model. Estimation results are shown in table 2. Although each forecasted daily volatility is generated by model which is estimated by fitting data from previous 735 transaction days, the estimation results obtained

from fitting the whole sample give us some insight on how good data are fitted into these models.

Table 2: Estimation results by fitting data from 15 February 1999 to 20 April 2004. Estimation results include estimated parameters of GARCH models. Standard errors are in parenthesis. Akaike information criterion (AIC) is calculated as  $-2(\ln L)+2p$  where L is the likelihood function and p is the number of parameters being estimated.

	GARCH model			
		with historical volatility	with realized volatility	with implied volatility
$\hat{\mu}$	0.0425 (0.0440)	-0.0535 (0.0834)	-0.0295 (0.0608)	-0.1325 (0.0981)
$\hat{\alpha}$	0.0422 (0.0095)	0.0430 (0.0097)	0.0436 (0.0097)	0.0442 (0.0097)
$\hat{\beta}$	0.9460 (0.0124)	0.9445 (0.0128)	0.9436 (0.0128)	0.9426 (0.0130)
$\hat{\omega}$	0.0353 (0.0146)	0.0373 (0.0153)	0.0380 (0.0155)	0.0395 (0.0159)
$\hat{\gamma}$		0.0396 (0.0297)	0.0329 (0.0168)	0.0748 (0.0336)
AIC	4773.23	4768.61	4767.07	4765.92

For the simple GARCH model, all parameters are statistically significant except the mean  $\hat{\mu}$ . Note that whether  $\hat{\mu}$  is statistically different from 0 does not affect the adequacy of the GARCH model. In addition,  $\hat{\alpha} + \hat{\beta} = 0.9882$ , which is close to unity, shows a large degree of volatility persistency.

When historical volatility is added to the GARCH model as an explanatory variable, volatility persistency is still statistically significant as  $\hat{\alpha} + \hat{\beta} = 0.9875$ . Although the estimated regression coefficient  $\hat{\gamma}$  is not statistically significant at 5% significance level, the AIC value decreases from 4773 to 4768. It is not surprising that the coefficient  $\hat{\gamma}$  is not statistically significant because inclusion of historical volatility in the GARCH model does not add much additional information to the original GARCH model. Historical volatility is constructed from daily return and the information extracted from daily return has already been captured by the simple GARCH model. Based on estimation results only, it remains questionable whether GARCH model with historical volatility as explanatory variable is adequate for volatility forecasting.

Inclusion of implied volatility in the GARCH model has little effect on the fit of model. Volatility persistency continues to be statistically significant and the AIC value decreases from 4773 to 4765. The p-value for testing significance of parameter  $\hat{\gamma}$  is 0.0261, indicating that inclusion of implied volatility helps improve model fitting, but not to a great extent. No estimation result goes against the inclusion of implied volatility as explanatory variable.

When realized volatility is added to the GARCH model as an explanatory variable, the estimation results are not as good as that of implied volatility. Although the AIC value decreases from 4773 to 4767, the regression coefficient  $\gamma$  is only marginally significant as it has a p-value close to 5%. Volatility persistency is statistically significant and close to unity as  $\hat{\alpha} + \hat{\beta} = 0.9872$ .

## 5.2 Forecasting Results

The forecasting methodology is described in section 4.1. The evaluation period is from 15 February 2002 to 20 April 2004, which consists of 500 transaction days. For each transaction day in this evaluation period, a daily volatility forecast is generated by the GARCH model which is estimated by fitting data from the previous 735 transaction days. In figure 5, daily return series and daily volatility forecast series are plotted.

The forecasted daily volatilities generated by the simple GARCH model is similar to those generated by GARCH models with either historical volatility, realized volatility or implied volatility as explanatory variable. This is within our expectations. Based on estimation results in section 5.1, inclusion of either historical volatility, realized volatility or implied volatility in GARCH model only improves little on model fitting.

GARCH models fail to respond to sudden great changes in daily return volatility immediately. As demonstrated in figure 5, any sudden great increase in daily return volatility is followed by increased values in forecasted daily volatility a several days later. This cannot lead us to the conclusion that forecasted daily volatilities are not accurate. Great increase in daily return volatility could be caused by other environmental factors, which cannot be predicted by GARCH models.

With reference to table 2,  $\hat{\mu}$  has large standard error and is statistically insignificant from 0 for all GARCH models. Including it in equation (14) tends to reject  $H_0$  in section 4.2. We therefore assume daily returns to be normally distributed with mean 0. Condition (14) becomes

$$R_n < -1.645\hat{\sigma}_n \quad (15)$$

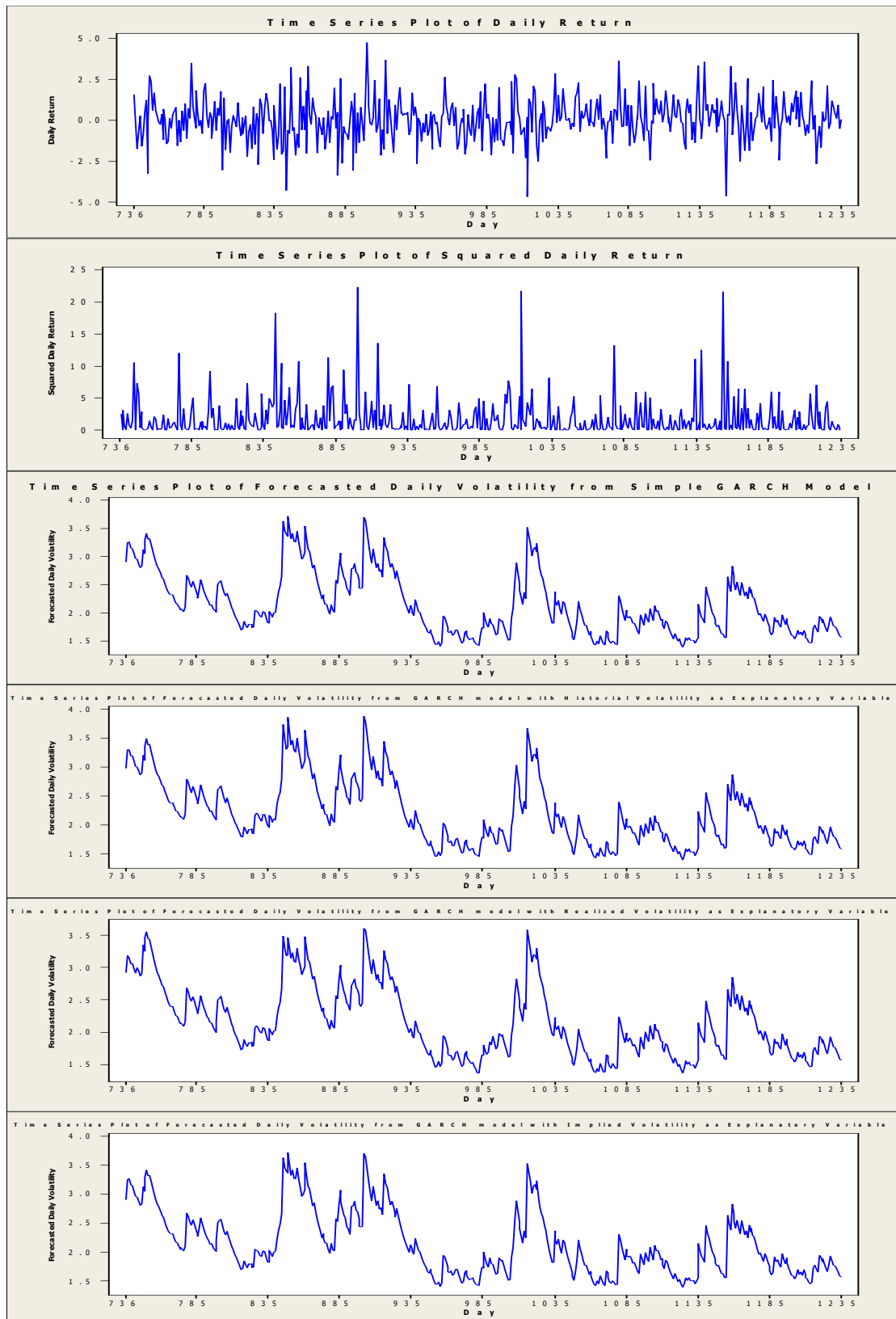


Figure 5: Time series plot of daily return  $R_n$  (first row), squared daily return  $R_n^2$  (second row), forecasted volatility from simple GARCH model (third row), GARCH model with historical volatility as explanatory variable (fourth row), GARCH model with realized volatility as explanatory variable (fifth row) and GARCH model with implied volatility as explanatory variable (sixth row) from day 736 to day 1235

For the simple GARCH model, out of the 500 returns in the evaluation period, only 16 of them satisfy condition (15). Although we marginally fail to reject  $H_0$  at 5% significance level, we cannot conclude that forecasted volatilities from simple GARCH model are very accurate because its p-value is close to 5%.

For the GARCH model with historical volatility as explanatory variable, out of the 500 returns in the evaluation period, 17 of them satisfy condition (15). Since 17 is greater than 15.4 and smaller than 34.6, we can conclude that forecasted daily volatilities are accurate in this model, at 5% significance level. Forecasting performance improves a little bit when compared to that of the simple GARCH model. This is consistent with the estimation results in section 5.1, which suggest that inclusion of historical volatility only adds little information to the model.

For the GARCH model with realized volatility as explanatory variable, out of the 500 returns in the evaluation period, 18 of them satisfy condition (15). Inclusion of realized volatility in the GARCH model has some improvement in forecasting performance. It leads us to the conclusion that forecasted daily volatilities in this model are accurate, at 5% significance level.

For the GARCH model with implied volatility as explanatory variable, out of the 500 returns in the evaluation period, 18 of them satisfy condition (15). The result and conclusion are exactly the same as that of GARCH model with realized volatility as explanatory variable.

If we compare the models that have been discussed so far, inclusion of either historical, realized or implied volatility in simple GARCH model as explanatory variable improves a little on model forecasting. At 5% significance level, we fail to reject the hypothesis that forecasted daily volatilities are accurate for all models we have discussed so far.

Table 3: p-values for binomial test

	GARCH model			
	Simple	With Historical Volatility	With Realized Volatility	With Implied Volatility
p-value	0.0648	0.1007	0.1509	0.1509

## 6 Reserving by Simulation

As discussed earlier, forecasted Hang Seng Index daily volatilities from the GARCH model could be used to predict the most adverse Hang Seng Index price with a defined time frame and pre-defined level of confidence by simulation, which could then be applied to reserving for investment guarantees. The following example demonstrates how reserving is estimated by simulation.

Suppose today is day 735 and we are predicting the most adverse Hang Seng Index price at day 1000 with 95% level of confidence. Recall in section 4.2 that daily return at day  $n$ ,  $R_n$ , is assumed to be normally distributed with mean  $\mu_n$  and variance  $\sigma_n^2$ . First, 265 random numbers between (0,1) are generated by a random generator and considered to be the cumulative distribution function of returns for each day respectively from day 736 to day 1000. Next, mean of return  $\mu_n$  is assumed to be the same from day 1 to 1000 in the GARCH model and is the estimated parameter  $\hat{\mu}$  by fitting data from day 1 to day 735. Daily volatility for day 736 are forecasted based on data from day 1 to day 735. As we have the cumulative distribution function, estimated mean and forecasted daily volatility for day 736, we can determine daily return for day 736, hence the Hang Seng Index price at day 736. Similarly, Hang Seng Index price at day 1000 is determined by the simulated cumulative distribution functions from day 736 to day 1000, estimated mean and forecasted daily volatilities

based on data from day 1 to day 735 and simulated returns from day 736 to day 1000. If we perform the above simulation exercise 2000 times, we can simulate 2000 Hang Seng Index prices at day 1000. The most adverse Hang Seng Index price at day 1000 with 95% level of confidence is the 101<sup>st</sup> simulated price in ascending order among those 2000 simulated prices.

Most adverse Hang Seng Index price at different days with different levels of confidence could be determined by varying the simulation length and choosing different ranked simulated price respectively. However, as suggested by the CIA Task Force on Segregated Fund Investment Guarantees (2002), historical data should cover at least two times the projection span. Since Hang Seng Index prices are only available for years after 1969, any guaranteed product that matures in more than 18 years may not have sufficient historical data for projection and adjustments may be required.

When reserving for Hang Seng Index embedded investment guarantees at a pre-defined level of confidence, reserve could be set as the difference between the guaranteed value at maturity and the most adverse Hang Seng Index price at maturity with pre-defined level of confidence.

## 7 Conclusion and Further Research

In this study we examine the daily volatility forecasting abilities of GARCH model using historical, realized and implied volatility measurements. It is concluded that forecasting performances of GARCH model with different explanatory variables are satisfactory. Inclusion of historical, realized or implied volatility in the simple GARCH model yields improvements in modeling and forecasting, implying those three kinds of volatility measurements have some value in volatility forecasting.

As mentioned in section 6, any guaranteed product that matures in more than 18 years may not have sufficient historical data for projection and adjustments may be required. Future research directions include investigating how adjustments shall be made and the impact of those adjustments on projection. In addition, in view of the considerable materials and studies included in this paper, calibration test for the models, as suggested by the CIA Task Force on Segregated Fund Investment Guarantees (2002), and accuracy of the most adverse Hang Seng Index price with a defined time frame and pre-defined level of confidence are not validated with historical data. Further empirical studies may focus on developing calibration test for the models and investigating the accuracy of the most adverse Hang Seng Index price with historical data.

This study may have shed some light on the discussion of reserving for investment guarantees. While there is always room for improvement in the volatility forecasting approach, the authors believe that the paper is a good starting point in addressing the needs of insurers writing investment guarantees of the long term business in Hong Kong.

## References

Actuarial Society of Hong Kong (2001), Process for Determining Liabilities under the Guidance Note on Reserving Standards for Investment Guarantees as Issued by the Office of the Commissioner of Insurance.

(<http://www.actuaries.org.hk/doc/AGN%208.doc>)

American Academy of Actuaries (2001), Report of the VAGLB Work Group to the NAIC Life and Health Actuarial Task Force Nashville.

([www.actuary.org/pdf/life/naic/docs\\_vaglb\\_040201.pdf](http://www.actuary.org/pdf/life/naic/docs_vaglb_040201.pdf))

- Barndorff-Nielsen, O.E. and N. Shephard (2002), Econometric analysis of realized volatility and its use in estimating Stochastic Volatility models, *Journal of the Royal Statistical Society, Series B* **64**, 253-280.
- Bollerslev, T. (1986), Generalized autoregressive conditional heteroskedasticity, *Journal of Econometrics* **31**, 307-327.
- Canadian Institute of Actuaries (2002), CIA Task Force on Segregated Fund Investment Guarantees.  
(<http://www.actuaries.ca/members/publications/2002/202012e.pdf>)
- Christensen, G.J. and N.R. Prabhala (1998), The relation between implied and realized volatility, *Journal of Financial Economics* **50**, 125-150.
- Engle, Robert F (1982), Autoregressive conditional heteroskedasticity with estimates of the variance of the United Kingdom Inflation, *Econometrica* **50**, 987-1007.
- Fleming, J. (1998), The quality of market volatility forecast implied by S&P 100 index option prices, *Journal of Empirical Finance* **5**, 317-345.
- Hol, E., B. Jungbacker, and S.J. Koopman (2005), Forecasting daily variability of the S&P 100 stock index using historical, realized and implied volatility measurements, *Journal of Empirical Finance* **12**, 445-475.
- Martens, M. (2001), Forecasting daily exchange rate volatility using intraday returns, *Journal of International Money and Finance* **20**, 1-23.
- Shephard, N. (1996), Statistical aspects of ARCH and Stochastic Volatility, In D.R. Cox, D.V. Hinkley, and O.E. Barndorff-Nielsen (Eds.), *Time Series Models in Econometrics, Finance and Other Fields*, Number 65 in Monographs on Statistics and Applied Probability, pp. 1-67, Chapman and Hall, London.